Soliton Model of Hydrogen Atom: Resonance Effects

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Some first principles that, we believe, could serve as foundation for quantum theory of extended particles are formulated. It is also shown that in the point-like particles limit the non-relativistic quantum mechanics can be restored. As an illustration the soliton model of hydrogen atom is considered.

To begin with we formulate the first principles for quantum theory of extended particles:

- Following A. Einstein and L. de Broglie we describe the extended particles by the stable soliton-like solutions to non-linear field equations.
- Along the line of D. Bohm's thought we accept that the wave properties of particles have the origin in non-linear resonance effect.
- We assume that the statistical properties of particles can be deduced in the point-like limit from an analog of the wave function describing the quantum statistical ensemble of D. Blokhintsev.

To illustrate these principles we consider the simplest scalar field model given by the Lagrangian in the Minkowski space-time

$$L_0 = \partial_i \phi^* \partial_j \phi \eta^{ij} - (mc/\hbar)^2 \phi^* \phi + F(S), \quad S = \phi^* \phi, \tag{1}$$

with $F(S)$ behaving as S^n, $n > 1$, for $S \to 0$. This model admits, for many choices of F, e.g., $F = kS^n, \ k > 0, \ 1 < n < 5/3$, stable soliton-like solution of stationary type

$$\phi_0 = u(r) e^{-i\omega_0 t}, \quad r = |\mathbf{r}|, \tag{2}$$

with the energy

$$E = \int d^3x T^{00}(\phi_0) \tag{3}$$

and the electric charge

$$Q = e\omega_0 \int d^3x |u|^2. \tag{4}$$

D. Bohm in his book "Causality and Chance in Modern Physics" (1957) discussed the following problem. Let $\phi = \phi_0 + \xi(t, \mathbf{r})$ describes the perturbed soliton-like solution. D. Bohm put the following question: Does there exist any nonlinear model for which the spatial asymptote of $\xi(r \to \infty)$ represents oscillations with characteristic frequency

$$\omega = E/\hbar? \tag{5}$$

As is clear from the structure of the Lagrangian (1), at spatial infinity the field equation reduces to the linear Klein-Gordon one

$$[\Box - (mc/\hbar)^2]\phi = 0, \tag{6}$$

and therefore the principle of non-linear resonance by Bohm (5) holds only for solitons with the energy $E = mc^2$. It that the universality of the Planck-de Broglie relation (5) fails. To reinstate the universality of the relation (5) we modify the model (1) including gravity:

$$L = c^4 R/16\pi G + \partial_i \phi^* \partial_j \phi \eta^{ij} - I(g_{ij})\phi^* \phi + F(\phi^* \phi). \tag{7}$$

The crucial point of the model is to choose the invariant $I(g_{ij})$ with the asymptotic property

$$\lim_{r\to\infty} I(g_{ij}) = (mc/\hbar)^2, \tag{8}$$

where m stands for the Schwarzschild mass of the soliton. It can be verified that the relation (8) holds if one chooses

$$I = (I_1^4/I_2^3)c^6\hbar^{-2}G^{-2}, \tag{9}$$

where $I_1 = R_{ijkl}R^{ijkl}/48$, $I_2 = -R_{ijkl;n}R^{ijkl;n}/432$. Estimating R^{ijkl} at large distance one finds $I_1 = G^2m^2/(c^4r^6)$, $I_2 = G^2m^2/(c^4r^8)$. Thus we conclude that the principle of wave-particle duality has the gravitational origin in our model [1]. Now let us construct the analog of the wave function. Suppose that the field ϕ describes n particles and has the form

$$\phi(t, \mathbf{r}) = \sum_{k=1}^{n} \phi^{(k)}(t, \mathbf{r}), \tag{10}$$

where

$$\mathrm{supp}\,\phi^{(k)} \cap \mathrm{supp}\,\phi^{(k')} = 0, \quad k \neq k',$$

and the same for the conjugate momenta

$$\pi(t, \mathbf{r}) = \partial L/\partial \phi_t = \sum_{k=1}^{n} \pi^{(k)}(t, \mathbf{r}), \quad \phi_t = \partial\phi/\partial t.$$

Let us define the auxiliary functions

$$\varphi^{(k)}(t, \mathbf{r}) = \frac{1}{\sqrt{2}}(\nu_k \phi^{(k)} + i\pi^{(k)}/\nu_k) \tag{11}$$

with the constants ν_k satisfying the normalization condition

$$\hbar = \int d^3x |\phi^{(k)}|^2. \tag{12}$$

Now we define the analog of the wave function in the configurational space $\{\mathbf{r}_1, \cdots \mathbf{r}_n\} \in \Re^{3n}$ as

$$\Psi_N(t, \mathbf{r}_1, \cdots \mathbf{r}_n) = (\hbar^n N)^{-1/2} \sum_{i=1}^{N} \prod_{k=1}^{n} \varphi_i^{(k)}(t, \mathbf{r}_k), \tag{13}$$

where $N \gg 1$ stands for the number of trials (observations) and $\varphi_i^{(k)}$ is the one-particle function (11) for the i -th trial. It can be shown [1] that the quantity

$$\rho_N = \frac{1}{(\triangle\vee)^n} \int\limits_{(\triangle\vee)^n \subset \Re^{3n}} d^{3n}x |\Psi_N|^2,$$

where $\triangle\vee$ is the elementary volume which is supposed to be much greater than the proper volume of the particle $\vee_0 \ll \triangle\vee$, plays the role of coordinate probability density. If we choose the classical observable A with the generator $\hat{M}_A$, one can represent it in the form

$$A_j = \int d^3x \pi_j i \hat{M}_A \phi_j = \sum_{k=1}^{n} \int d^3x \varphi_j^{*(k)} \hat{M}_A^{(k)} \varphi_j^{(k)}, \tag{14}$$

for the j - th trial. The corresponding mean value is

$$
\begin{aligned}
< A > &= \frac{1}{N}\sum_{j=1}^{N} A_j = \frac{1}{N}\sum_{j=1}^{N}\sum_{k=1}^{n}\int d^3x \varphi_j^{*(k)} \hat{M}_A^{(k)} \varphi_j^{(k)} \\
&= \int d^3x \Psi_N^* \hat{A} \Psi_N + O(\frac{\vee_0}{\triangle\vee})
\end{aligned} \tag{15}
$$

where the hermitian operator $\hat{A}$ reads

$$
\hat{A} = \sum_{k=1}^{n} \hbar \hat{M}_A^{(k)}. \tag{16}
$$

Thus, upto the terms of the order $\vee_0/\triangle\vee \ll 1$, we obtain the standard quantum mechanical rule for the calculation of mean values [1]. It is interesting to underline that the solitonian scheme contains also the well-known spin - statistic correlation [1]. Namely, if $\varphi_i^{(k)}$ is transformed under the group rotation by irreducible representation $D^{(J)}$ of $SO(3)$, then the transposition of two identical extended particles is equivalent to the relative 2π rotation of $\varphi_i^{(k)}$ that gives the multiplication factor $(-1)^{2J}$ in Ψ_N. It can be also proved that Ψ_N upto the terms of order $\vee_0/\triangle\vee$ satisfies the standard Schrödinger equation [1]. Now we apply the solitonian scheme to the hydrogen atom [2]. Let us introduce the nucleus Coulomb field $A_i^{\text{ext}} = \delta_i^0 Ze/r$ and consider the scalar field Lagrangian density

$$
\mathcal{L} = -\frac{1}{16\pi}(F_{ik})^2 + |[\partial_k - i\varepsilon(A_k + A_k^{\text{ext}})]\phi|^2 - (mc/\hbar)^2\phi^*\phi + F(\phi^*\phi), \tag{17}
$$

where $\varepsilon = e/\hbar c$. Suppose that for $A_k^{\text{ext}} = 0$ the field equations admit stable stationary soliton-like solution of type (2) describing configurations with mass m and electric charge e. For simplicity we omit the gravitational field supposing that it has been taken into account due to the non-linear resonance condition (5). Then, in the non-relativistic approximation we may put

$$
\phi = \psi \, \exp(-imc^2 t/\hbar). \tag{18}
$$

Therefore, the corresponding field equations read

$$
\begin{aligned}
& i\hbar\,\partial_t\psi + (\hbar^2/2m)\triangle\psi + (Ze^2/r)\psi = -(\hbar^2/2m)\hat{f}(\mathbf{A}, A_0, \psi^*\psi)\psi \\
& \quad \equiv -(\hbar^2/2m)\Big[2i\varepsilon(\mathbf{A}\nabla)\psi + 2(\varepsilon mc/\hbar)A_0\psi + i\varepsilon\psi\,\mathrm{div}\mathbf{A} + F'(\psi^*\psi)\psi\Big], && (19) \\
& \Box A_0 = (8\pi me/\hbar^2)|\psi|^2 \equiv -4\pi\varrho, && (20) \\
& \Box\mathbf{A} = 4\pi[2\varepsilon^2\mathbf{A}|\psi|^2 - i\varepsilon(\psi^*\nabla\psi - \psi\nabla\psi^*)] \equiv -(4\pi/c)\,\mathbf{j}, && (21) \\
& \partial_t A_0 + c\,\mathrm{div}\mathbf{A} = 0 && (22)
\end{aligned}
$$

We will seek for the solutions to these equations describing a stationary state of an atom when the electron - soliton center moves along a circular orbit of radius a_0 with some angular velocity Ω. We have two characteristic lengths in this problem: the size of the soliton $\ell_0 = \hbar/mc$ and the Bohr radius $a = \hbar^2/mZe^2 \gg \ell_0$. Near the soliton center, where $r - a_0 \leq \ell_0$, we get in non-relativistic approximation

$$
\psi = u(\mathbf{R})e^{iS/\hbar} = \psi_-, \quad A_0 = A_0(\mathbf{R}), \quad \mathbf{A} = \frac{1}{c}\dot{\xi}(t)A_0(\mathbf{R})
$$

with

$$
\begin{aligned}
& S \approx m\dot{\xi}\cdot\mathbf{R} + C_0 t + \chi(t), \quad m\ddot{\xi} = -Ze^2\xi/\xi^3, \\
& \chi(t) = \int_0^t (\frac{m}{2}\dot{\xi}^2 + \frac{Ze^2}{\xi})dt - \quad \text{the Hamiltonian action.}
\end{aligned}
$$

The function $u(\mathbf{R})$, where $\mathbf{R} = \mathbf{r} - \xi(t)$ satisfies the following soliton-like equation $\hbar^2(\hat{f} + \triangle u/u) = 2mC_0$. For ψ we have the integral equation

$$
\begin{aligned}
\psi(t, \mathbf{r}) &= C_n\psi_n(\mathbf{r})\,\exp(-i\omega_n t) \\
&+ \frac{1}{2\pi}\int d\omega \int dt' \int d^3x' \,\exp[-i\omega(t - t')]\,G(\mathbf{r}, \mathbf{r}'; \omega + i0)\hat{f}\psi(t', \mathbf{r}'),
\end{aligned} \tag{23}
$$

with G being the Coulomb resolvent, $E_n = \hbar\omega_n$ is the eigenvalue of the Coulomb Hamiltonian. For $R \gg \ell_0$ we may put in (23)

$$\hat{f}\,\psi(t,\mathbf{r}) = g\,\exp\left(-i\omega_n t\right)\delta(\mathbf{r}-\xi(t)), \quad g = const.$$

Calculating the integral (23) by stationary phase method we get

$$\psi = \psi_+ \approx C_n\psi_n(\mathbf{r})\,e^{-i\omega_n t} - \frac{g|\omega_n|ma}{8\pi^2\hbar\sqrt{a_0\cos^2(\vartheta/2)}}e^{-i\omega_n t}R^{-3/2}e^{-R\sqrt{2m|\omega_n|/\hbar}},$$

where $\cos\vartheta = \sin\theta\cos(\alpha-\Omega t)$. Now to find the constants C_0, C_n, a_0, Ω, g we must match the functions ψ_+ and ψ_- at $R = \ell_0$. That gives the following results

$$a_0 = an, \quad \Omega^2 = Ze^2/ma_0^3, \quad C_0 = -m\Omega^2 a_0^2,$$
$$C_n\psi_n(a_0) = \frac{g|\omega_n|ma}{8\pi^2\hbar\sqrt{a_0}}\ell_0^{-3/2}e^{-\ell_0\sqrt{2m|\omega_n|/\hbar}} + u(\ell_0),$$
$$g = \int\limits_{\vee_0} d^3x\hat{f}u, \quad \vee_0 = \frac{4}{3}\pi\ell_0^3.$$

The last step is the calculation of the electromagnetic field for $R \gg \ell_0$ and for large time $t \gg 1/|\omega_n|$, that gives the semi-sum of the retarded and advanced potentials: $A_\mu = \frac{1}{2}\left(A_\mu^{\rm adv} + A_\mu^{\rm ret}\right)$. It is interesting to write down the components of the Poynting vector $\mathbf{S}$:

$$S_r = \frac{e^2\,a_0^2\,\Omega^4}{16\pi c^3\,r^2}\sin^2\vartheta\,\sin2(\alpha-\Omega t)\sin(2\Omega r/c),$$
$$S_\vartheta = \frac{e^2\,a_0\,\Omega^2}{4\pi c\,r^3}\cos\vartheta\,\sin(\alpha-\Omega t)\sin(\Omega r/c),$$
$$S_\alpha = \frac{e^2\,a_0\,\Omega^2}{4\pi c\,r^3}\cos(\alpha-\Omega t)\sin(\Omega r/c).$$

Thus we conclude that the radiation is absent. The various aspects of the solitonian scheme were discussed in details in [1,2].

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[2] Yu.P. Rybakov, and B. Saha, *Found. Phys.*, **25** (12), 1723 (1995); *Phys. Lett.*, **A 122** (1), 5 (1996).